To be more specific, the "momentum" operators $Q_j f = i^{-1}\partial f/\partial x_j$ and the "position" operators $Q_j f(x) = x_j f(x)$ on $L^2(\mathbb{R}^n)$ generate a Lie algebra called the Heisenberg algebra. The Heisenberg group $\mathbf{H}_n$ is the corresponding Lie group, and the exponentiated operators $\pi(p,q) = \exp\, i\Sigma(p_j P_j + q_j Q_j)$ $(p, q \in \mathbb{R}^n)$ determine a unitary representation of $\mathbf{H}_n$. The Gaussian function $e^{-\|x\|^2/2}$ on $\mathbb{R}^n$ plays a special role in this situation for several reasons—for example, it is the ground state of the harmonic oscillator and the extremal function for the uncertainty inequality—and hence the Hermite basis $\{h_\mu\}$ derived from it is of particular interest. Moreover, the matrix elements of π in this basis, $\Phi_{\mu\nu}(p,q) = \langle \pi(p,q)h_\mu, h_\nu\rangle$, turn out to give another orthonormal basis for $L^2(\mathbb{R}^{2n})$. The functions $\Phi_{\mu\nu}$ can be neatly expressed in terms of Laguerre functions, and they are also closely related to Hermite functions on $\mathbb{R}^{2n}$ since they are eigenfunctions for the Hermite operator, so they have been dubbed "special Hermite functions." The interplay among all these things is what gives this subject its distinctive flavor.

Somewhat more than half of the book is devoted to the detailed study of Hermite expansions and special Hermite expansions; the results are then used together with some transplantation theorems to derive information about Laguerre expansions. Much of this is an exposition of the author's own research. It provides an intriguing display of the concepts and techniques of modern Fourier analysis in a setting that has deep classical roots but still deserves further exploration.

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C. DE BOOR, K. HÖLLIG, AND S. D. RIEMENSCHNEIDER, *Box Splines*, Applied Mathematical Sciences, Vol. **98**, Springer-Verlag, 1993, xvii + 200 pp.

Box splines are a natural extension of cardinal splines. A box spline is a compactly supported smooth piecewise polynomial function. Such functions provide an efficient tool for the approximation of curves and surfaces and other smooth functions. Box splines give rise to a beautiful mathematical theory that is much richer than the univariate case. The richness of the box spline theory is due to the complexity of constructing smooth piecewise polynomials on polyhedral cells. On the other hand, it is this richness that allows box splines to have widespread applications.

The book under review, written by three pioneering mathematicians in this area, is the first book to give a complete account of the basic theory of box splines. The authors have not only organized the available material in a cohesive way, but also provide simple and complete proofs in many cases. A large number of illustrations in the book makes its reading enjoyable.

The book begins with a comprehensive description of the box spline and then discusses its various aspects. Given an $s \times n$ real matrix Ξ, the box spline M_Ξ associated with Ξ is defined to be the distribution given by the rule $\phi \mapsto \int_{[0,1)^n} \phi(\Xi t)\, dt$ for $\phi \in C(\mathbb{R}^s)$. In Chapter I various equivalent definitions of the box spline are given. Basic properties of the box spline are derived, and several detailed examples and illustrations are provided to support the mathematical discussion.

The main body of the book falls into three categories: (1) algebraic theory of box splines; (2) approximation and interpolation by box splines, and (3) wavelets and subdivision schemes induced by box splines.

The algebraic theory of box splines is established in Chapter II and further developed in Chapter VI. The box spline M_Ξ is a piecewise polynomial function. These polynomials form a finite dimensional space $D(\Xi)$. The space $D(\Xi)$ can be described as the joint kernel of certain linear partial differential operators with constant coefficients. If each differential operator is replaced by its corresponding difference operator, then the joint kernel $\Delta(\Xi)$ of the resulting difference operators is a linear space of sequences on $\mathbb{Z}^s$. The structure of

$D(\Xi)$ and $\Delta(\Xi)$ is carefully examined in Chapter II. In particular, the dimensions of $D(\Xi)$ and $\Delta(\Xi)$ are computed and a standard basis for each of $D(\Xi)$ and $\Delta(\Xi)$ is constructed. These results are applied to the study of linear independence of the integer translates of a box spline. For instance, it is shown that the integer translates of M_Ξ are linearly independent if and only if $\Delta(\Xi)$ is the restriction of $D(\Xi)$ to $\mathbb{Z}^s$. The space $\Delta(\Xi)$ is also crucial to the study of discrete box splines. In Chapter VI, the theory of discrete box splines is developed in close analogy to that of the (continuous) box spline. In particular, the local structure of the discrete box spline is discussed. Also, the linear independence of the integer translates of a discrete box spline is characterized in terms of its defining matrix. This work is a beautiful application of linear diophantine equations. It is perhaps surprising that discrete box splines can also be used to study linear diophantine equations and solve some difficult combinatorial problems.

Chapters III, IV, and V of the book are devoted to approximation and interpolation by box splines. Given a box spline $M = M_\Xi$, the cardinal spline space $S = S_M$ consists of all infinite linear combinations of the integer translates of M. The space S gives rise to the scale $(S_h)_{h>0}$, where $S_h := \{g(\cdot/h): g \in S\}$. The approximation order of S is defined to be the largest r for which $\operatorname{dist}(f, S_h) = O(h^r)$ for all sufficiently smooth f, with the distance measured in the $L_p(G)$-norm ($1 \le p \le \infty$) on some bounded domain G. In Chapter III it is shown that the approximation power of S is determined by the linear space of all polynomials contained in S, which in turn coincides with D_Ξ. The authors demonstrate that the optimal approximation order of S can be attained by a quasi-interpolant, and construct the desired quasi-interpolant via Fourier transform and via Neumann series. Chapter IV deals with cardinal interpolation, i.e., the interpolation to data on the integer mesh, from the cardinal spline space S_M. The cardinal interpolation with M is said to be correct if for any bounded sequence there exists a unique bounded interpolant. It is proved that the cardinal interpolation with M is correct if and only if its symbol $\widetilde{M}$ does not vanish, where $\widetilde{M}(y) := \sum_{j\in\mathbb{Z}^s} M(j)\exp(-ijy)$ for $y \in \mathbb{R}^s$. On the basis of this result, the authors show that the cardinal interpolation with any centered box spline associated with three directions in $\mathbb{R}^2$ is correct. Furthermore, the authors in Chapter V investigate the approximation of cardinal interpolants as the degree of the box spline tends to infinity, or as the mesh width goes to zero.

A box spline M is refinable. More precisely, if $1/h$ is a positive integer, then there exists a finitely supported sequence m^h such that $M = \sum_{j\in\mathbb{Z}^s} m^h(j)M(\cdot/h - j)$. This equation is called a refinement equation and m^h is called the refinement mask. A repeated application of the refinement equation can be used to produce the box spline, or more generally, any box spline surface of the form $\sum_{k\in\mathbb{Z}^s} M(\cdot - k)a(k)$. Such a process is called a subdivision algorithm. In Chapter VII, the authors discuss linear convergence and quadratic convergence of the subdivision algorithm. This chapter contains some new results published for the first time. The study of subdivision algorithms for box splines preceded the current research in more general refinement equations, or two-scale dilation equations, which play an important role in wavelet analysis. In Chapter V, the authors include a brief introduction to wavelet analysis. Let $S = S_M$ be the cardinal spline space associated with a box spline M, and let S_k denote the 2^k-dilate of S. It is shown in Chapter V that $S_k (k \in \mathbb{Z})$ form a multiresolution of $L_2(\mathbb{R}^s)$. Wavelet decompositions are then discussed. Of particular interest is the construction of wavelets in $\mathbb{R}^s$ for $s \le 3$. This construction not only applies to box splines, but also applies to a wide class of functions in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In summary, this is a clearly written research monograph that lays a foundation for box spline theory. It will definitely be a welcome addition to the library of researchers in analysis, numerical analysis, and engineering. It can also be used as a textbook for a graduate-level course.

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